Differential Essential Dimension

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Assumption

F is a field of characteristic zero.

How much can we simplify polynomials via Tschirnhaus transformations over F?

$$p(x) = x^2 + ax + b \quad \overset{x=y-a/2}{\rightsquigarrow} \quad q(y) = y^2 + c \quad (c = b - a^2/4)$$

$$p(x) = x^3 + ax^2 + bx + c \quad \overset{x=y-a/3}{\rightsquigarrow} \quad q(y) = y^3 + dy + e$$

$$\overset{y=(e/d)z}{\rightsquigarrow} \quad r(z) = z^3 + fz + f$$

Consider the general polynomial over F:

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad (a_i\text{'s algebraically independent over } F).$$

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Let $d(n)$ be the min number of algebraically independent coefficients of $q(y)$, as $q(y)$ ranges over the more general "Tschirnhaus transformations" of $p(x)$ over $F(a_0, \ldots, a_{n-1})$.

n	2	3	4	5	6	7
$d(n)$	1	1	(BR)	2 (Hermite and Klein)	3 (BR)	4 (A. Duncan, 2010)

J. Buhler and Z. Reichstein (1997) introduced essential dimension to prove:

$$d(4) = 2, \qquad \lfloor n/2 \rfloor \leq d(n) \leq n - 3 \quad (n \geq 5).$$

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Assumptions for the rest of the talk

F is a differential field, i.e. a field with a derivation $\partial : F \to F$ like $(F, \partial) = (\mathbb{C}(x), d/dx)$, and $\operatorname{char} F = 0$.

Its constant field $C = \{c \in F \mid \partial(c) = 0\}$ is algebraically closed and properly contained in F.

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How much can we simplify $n \times n$ matrix DEs?

An $n \times n$ matrix DE $Z' = BZ$ is a gauge transformation of $Y' = AY$ over F if $Z = PY$ for some $P \in \mathrm{GL}_n(F)$.

Consider the general matrix DE

$$Y' = AY \quad (A_{ij}\text{'s differentially independent over } F)$$

i.e. the matrix entries A_{ij} and their higher derivatives are algebraically independent over F.

Let $e(n)$ be the min number of differentially independent coefficients of $Z' = BZ$, as $Z' = BZ$ ranges over the gauge transformations of $Y' = AY$ over

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Can gauge transform $Y' = AY$ to some

$$Z' = BZ, \quad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{pmatrix}.$$

So $e(n) \leq n$.

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Where did this $Z' = BZ$ come from?

Homogeneous linear differential equation of order n

$$p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$$

corresponds to $n \times n$ matrix differential equation

$$Y' = A_p Y, \quad Y = \begin{pmatrix} y \\ y' \\ \cdots \\ y^{(n-1)} \end{pmatrix}, \quad A_p = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}$$

and also corresponds to differential module of rank n over F i.e. vector space of dimension n over F with a derivation

$$M = F \cdot y \oplus F \cdot y' \oplus \cdots \oplus F \cdot y^{(n-1)}, \qquad \begin{aligned} (y^{(i)})' &:= y^{(i+1)} \\ (y^{(n-1)})' &:= -a_0 y - a_1 y' - \cdots - a_{n-1} y^{(n-1)}. \end{aligned}$$

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Picard-Vessiot extensions

Differential torsors

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The Picard-Vessiot theory - “differential Galois theory"

Given:

- An $n \times n$ matrix DE $Y' = AY$ over F
- A differential field extension K of F again with constant field C
- A solution matrix $y \in \mathrm{GL}_n(K)$, i.e., $y' = Ay$

Then:

- The Picard-Vessiot ring of $Y' = AY$ is

$$R := F\left[y_{ij}, \tfrac{1}{\det(y)} \mid 1 \leq i, j \leq n\right].$$

- The differential Galois group of R/F,

$$\mathrm{Gal}^{\partial}(R/F) := \{\text{differential } F\text{-algebra automorphisms of } R\},$$

  is isomorphic to $G(C)$ for some closed subgroup G of $\mathrm{GL}_{n,C}$.

- We say R/F is a G-Picard-Vessiot extension.

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The Picard-Vessiot theory: examples over $F = \mathbb{C}(x)$

Example. (Exponential)

$Y' = Y$ has solution $y = e^x$ over $K = \mathbb{C}((x))$.

$$R = \mathbb{C}(x)[e^x, e^{-x}]$$

$$\Big| \ \mathbb{G}_m$$

$$F = \mathbb{C}(x)$$

Any differential ring automorphism must take the solution e^x to $y' = y$ in R to another solution ce^x of the same equation, with $c \neq 0$, so $\mathrm{Gal}^{\partial}(R/F) = \mathbb{C}^{\times} = \mathbb{G}_m(\mathbb{C})$.

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Example. (Logarithmic)

$$Y' = \begin{pmatrix} 0 & 1 \\ 0 & 1/x \end{pmatrix} Y \quad \text{has a solution matrix} \quad y = \begin{pmatrix} 1 & \log x \\ 0 & 1/x \end{pmatrix}$$

over $K = \mathbb{C}((x-1))$.

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Any differential ring automorphism takes the solution $\log x$ to $y' = 1/x$ in R to another solution $\log x + c$ of the same equation, so $\mathrm{Gal}^{\partial}(R/F) = \mathbb{C} = \mathbb{G}_a(\mathbb{C})$.

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Torsors in various categories (equipped with a Grothendieck topology)

G a group object in the appropriate category $\mathscr{C}$.

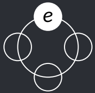
X an object in $\mathscr{C}$ with a right G-action $X \times G \to X : x \mapsto x.g$.

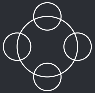
X is a G-torsor if

$$X \times G \to X \times X : (x, g) \mapsto (x, x.g)$$

is an isomorphism and X satisfies some "local triviality condition".

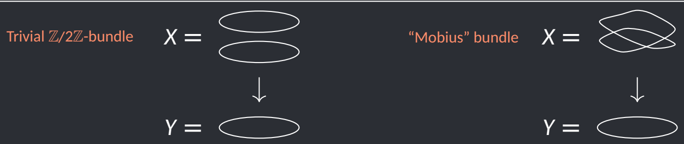
$\mathscr{C} =$ Sets. A G-torsor X is just the group G that forgot its identity but retains the G-action.

$G =$ 

$= \mathbb{Z}/4\mathbb{Z}$ acts on $X =$  by rotation.

Torsors in various categories (equipped with a Grothendieck topology)

$\mathscr{C} = \mathrm{Top}_Y$ for a topological space Y. A G-torsor X is a principal G-bundle, i.e., a continuous family of G's parametrized over Y.



$G = \mathbb{Z}/2\mathbb{Z}$ acts simply transitively on the fibers but there is no canonical way to identify each fiber with G

Torsors in various categories (equipped with a Grothendieck topology)

A Galois extension K/F with Galois group G satisfies

$$K \otimes_F K \cong \prod_G K \cong K \otimes_F F[G] \qquad \text{(by the normal basis theorem)}$$

where $F[G]$ is the coordinate ring of G when we view G as a finite constant group scheme over F. Thus $\mathrm{Spec}(K)$ is a G-torsor over $\mathrm{Spec}(F)$.

A G-Picard-Vessiot extension R/F satisfies

$$R \otimes_F R \cong R \otimes_C C[G] \qquad \text{as differential rings}$$

with $C[G]$ given the trivial derivation. Thus $\mathrm{Spec}(R)$ is a differential G-torsor over $\mathrm{Spec}(F)$.

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A summary

- Differential modules - intrinsic formulation of matrix differential equations
- Picard-Vessiot extensions - differential Galois extensions for matrix differential equations
- Differential torsors - Geometric formulation of Picard-Vessiot extensions

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Differential essential dimension

"How to count parameters"

Differential essential dimension ed^∂

The differential transcendence degree $\mathrm{trdeg}_F^\partial K$ is the size of biggest differentially independent subset of K over F.

Consider the following classes of objects:

- $\mathbf{DiffEq}_n(K) = \left\{ \substack{n \times n \text{ matrix DE} \\ \text{over } K \text{ up to gauge transformations}} \right\}$
- $\mathbf{Diff}_n(K) = \left\{ \substack{\text{(differential) isomorphism classes of} \\ \text{differential modules of rank } n \text{ over } K} \right\}$
- $G\textbf{-tors}^\partial(K) = \left\{ \substack{\text{(differential) isomorphism classes of} \\ \text{differential } G\text{-torsors over } K} \right\}$

Define:

- $\mathrm{ed}_F^\partial(a) := \min_K \mathrm{trdeg}_F^\partial K$ where K ranges over the differential fields K that some a' is defined over, where $a' \cong a$ in its class of objects.
- $\mathrm{ed}_F^\partial(\text{class of objects}) = \sup_{K,a} \mathrm{ed}_F^\partial(a)$, where K/F are differential fields whose constant field is C, and a/K are objects in that class.

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- $\mathrm{ed}_F^\partial(\text{class of objects}) = \sup_{K,a} \mathrm{ed}_F^\partial(a)$, where K/F are differential fields whose constant field is C, and a/K are objects in that class.

Differential essential dimension ed^∂

The differential transcendence degree $\mathrm{trdeg}_F^\partial K$ is the size of biggest differentially independent subset of K over F.

Consider the following classes of objects:

- **DiffEq**$_n(K) = \left\{ \substack{n\times n \text{ matrix DE} \\ \text{over } K \text{ up to gauge transformations}} \right\}$
- **Diff**$_n(K) = \left\{ \substack{\text{(differential) isomorphism classes of} \\ \text{differential modules of rank } n \text{ over } K} \right\}$
- G**-tors**$^\partial(K) = \left\{ \substack{\text{(differential) isomorphism classes of} \\ \text{differential } G\text{-torsors over } K} \right\}$

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Results on the differential essential dimension

- Saw: $\mathrm{ed}_F^\partial(\text{general } n \times n \text{ matrix DE}) = e(n) \leq n$.
- $\mathrm{ed}_F^\partial(G\text{-}\mathbf{tors}^\partial) = n$ for $G = (\mathbb{Z}/r\mathbb{Z})^n, \mathbb{G}_m^n, \mathbb{G}_a^n$.

  Intuition. By Kummer theory and the Kolchin-Ostrowski theorem, $(\mathbb{Z}/r\mathbb{Z})^n$-Galois extensions and $\mathbb{G}_m^n$- and $\mathbb{G}_a^n$-Picard-Vessiot extensions over K come from solving

  - $y_i^r = a_i \quad (i = 1, \ldots, n)$,
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Cohomology

Cohomology classifies various classes of objects

In topology:

$$\left\{\begin{matrix}\text{rank } n \text{ real vector bundles} \\ \text{over } Y \text{ up to iso}\end{matrix}\right\} \cong H^1\left(Y, \underline{\mathrm{GL}_n(\mathbb{R})}\right) \cong \left\{\begin{matrix}\text{principal } \mathrm{GL}_n\text{-bundles} \\ \text{over } Y \text{ up to iso}\end{matrix}\right\}$$

In Galois cohomology:

$$\left\{\begin{matrix}\text{quadratic forms up to} \\ \text{linear change of variables over } K\end{matrix}\right\} \cong H^1(K, \mathrm{O}_n) \cong \mathrm{O}_n\textbf{-tors}(K)$$

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central simple algebras — Severi-Brauer varieties

"Cohomology provides a bridge between different classes of objects."

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Cohomology classifies various classes of objects

We can construct a cohomology theory $H^1_\partial(K, \mathrm{GL}_n)$ analogous to Galois cohomology, giving:

$$\mathbf{DiffEq}_n(K) \cong \mathbf{Diff}_n(K) \cong H^1_\partial(K, \mathrm{GL}_n) \cong \mathrm{GL}_n\text{-}\mathbf{tors}^\partial(K)$$

Answer

General matrix differential equation

Under

$$\mathbf{DiffEq}_n(K) \cong \mathbf{Diff}_n(K) \cong H^1_\partial(K, \mathrm{GL}_n) \cong \mathrm{GL}_n\text{-}\mathbf{tors}^\partial(K),$$

one shows:

$$\text{general DE} \longmapsto \text{a "generic" differential } \mathrm{GL}_n\text{-torsor}$$

Therefore:

$$\begin{aligned} n \geq \mathrm{ed}^\partial_F(\text{general DE}) &= \mathrm{ed}^\partial_F(\text{this generic differential } \mathrm{GL}_n\text{-torsor}) \\ &= \mathrm{ed}^\partial_F(\mathrm{GL}_n\text{-}\mathbf{tors}^\partial) \\ &\geq \mathrm{ed}^\partial_F(\mathbb{G}_m^n\text{-}\mathbf{tors}^\partial) = n \end{aligned}$$

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"Any generic differential torsor is the most complicated in its class of objects."

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Subgroup bound.

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Computation.

General matrix differential equation

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$\Rightarrow e(n) = \mathrm{ed}^\partial(\text{general DE}) = n.$

Future directions

Simplifying differential equations with more general transformations

$$p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_0 y$$

$$\overset{y=e^{-\frac{1}{n}\int a_{n-1}}z}{\rightsquigarrow} \quad q(z) = z^{(n)} + 0 \cdot z^{(n-1)} + b_{n-2}z^{(n-2)} + \cdots + b_0 z$$

How many more parameters can we eliminate from the general differential equation if we allow more transformations like $\exp$ and $\int$?

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Thanks! Questions?

Extra slides

Cohomology

Γ a linear algebraic group over C.
G a linear algebraic group over C with compatible Γ-action.

A cocycle is a morphism of varieties $a : \Gamma(C) \to G(C) : \sigma \mapsto a_\sigma$ satisfying the usual cocycle condition in Galois cohomology.

Two cocycles a and b are equivalent if there exists $c \in \Gamma(C)$ such that $a_\sigma = c \cdot b_\sigma \cdot c^{-1}$.

$$H^1_\partial(\Gamma, G) = \{\text{cocycles up to equivalence}\}$$

$$H^1_\partial(F, G) = \varinjlim_{\substack{R/F \\ \text{Picard-Vessiot} \\ \text{extension}}} H^1_\partial(\mathrm{Gal}^\partial(R/F), G)$$

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